

# MOSAIC SUPERCONGRUENCES OF RAMANUJAN-TYPE

JESÚS GUILLERA

**ABSTRACT.** We generalize the patterns of supercongruences of Ramanujan-type observed by L. Van Hamme and W. Zudilin to series involving simple square roots anywhere and not only in the result of the sum. To support our observations we give some examples.

## 1. RAMANUJAN AND RAMANUJAN-SATO SERIES

The research of Srinivasa Ramanujan on elliptic integrals and modular equations lead him to discover 17 surprising series for  $1/\pi$  published in 1914. They are of the following form

$$\sum_{n=0}^{\infty} A_n(a + bn)z^n = \frac{1}{\pi},$$

where  $z$ ,  $a$  and  $b$  are algebraic numbers and

$$A_n = \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3}, \quad s = 1/2, 1/4, 1/3, 1/6.$$

During a long time these Ramanujan's series for  $1/\pi$  were almost ignored but since 1987 the interest of mathematicians in them was great. The 17 formulas as well as many other series of the same type are already proved rigorously [2]. Naturally, they are called Ramanujan-type series. In 2002 T. Sato surprised the mathematical community presenting a series like those of Ramanujan but involving the Apéry numbers

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Inspired in this result mathematicians discovered similar series involving other kinds of special numbers as the Domb numbers

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k},$$

the Almkvist-Zudilin numbers

$$A_n = \sum_{k=0}^n (-1)^{n-k} \frac{3^{n-3k} (3k)!}{(k!)^3} \binom{n}{3k} \binom{n+k}{k},$$

or others, like for example

$$A_n = \sum_{k=0}^n \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2, \quad A_n = \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3, \quad A_n = \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}.$$

The series involving these kind of numbers are called of Ramanujan-Sato-type and we will refer to these numbers  $A_n$  as Ramanujan-Sato-type sequences of numbers. We have taken the definitions from [1], [5], [6].

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*Key words and phrases.* Hypergeometric series; Supercongruences; Ramanujan-type series for  $1/\pi$  and  $1/\pi^2$ ; Ramanujan-Sato-type series.

## 2. MOSAIC SUPERCONGRUENCES

We generalize the patterns of supercongruences of Ramanujan-type observed by Van Hamme in [9] and W. Zudilin in [11] to series involving simple square roots anywhere and not only in the result of the sum.

**The new pattern.** Suppose  $A_n$  is a Ramanujan-Sato sequence of numbers,  $z, a, b$  are algebraic numbers, and

$$\sum_{n=0}^{\infty} A_n(a + bn)z^n = \frac{1}{\pi},$$

Suppose also that the partial sums belong to  $\mathbb{Q}(\sqrt{d_1}, \dots, \sqrt{d_j})$ , where  $d_1, \dots, d_j$  are square free integers, that is

$$\sum_{n=0}^{p-1} A_n(a + bn)z^n = \alpha_1(p)\sqrt{d_1} + \dots + \alpha_j(p)\sqrt{d_j},$$

where  $\alpha_1(p), \dots, \alpha_j(p)$  are rational and  $a = a_1\sqrt{d_1} + a_2\sqrt{d_2} + \dots + a_j\sqrt{d_j}$ . Then, for primes  $p > p_0$ , we have the supercongruences

$$\alpha_i(p) \equiv a_i \left( \frac{-d_i}{p} \right) p \pmod{p^3} \quad i = 1, 2, \dots, j.$$

We will refer to them as *mosaic supercongruences* because they are as pieces of a single sum expression. We recall that for some kinds of Ramanujan-Sato-type numbers the supercongruences hold only  $\pmod{p^2}$ .

For the Ramanujan-like series for  $1/\pi^2$ , discovered by the author, we conjecture analogue mosaic supercongruences again generalizing Zudilin's pattern. See the last two examples.

## 3. EXAMPLES

All the congruences in the following examples remain unproven.

**Example 1.** For the Ramanujan-type series

$$\frac{\sqrt{15}}{2^7 \cdot 5^2} \cdot \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (263 + 5418n) \frac{(-1)^n}{80^{3n}} = \frac{1}{\pi},$$

we have checked that if we write

$$\sqrt{15} \sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (263 + 5418n) \frac{(-1)^n}{80^{3n}} = \alpha_p \sqrt{15},$$

then, for primes  $p > 5$ , we have the following supercongruences

$$\alpha_p \equiv 263 \left( \frac{-15}{p} \right) p \pmod{p^3},$$

that is

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (263 + 5418n) \frac{(-1)^n}{80^{3n}} \equiv 263 \left( \frac{-15}{p} \right) p \pmod{p^3} \quad p > 5,$$

which is [11, eq. 21].

**Example 2.** For the Ramanujan-type series

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} \left( \frac{7\sqrt{7}-10}{27} + \frac{13\sqrt{7}-7}{9}n \right) \left( \frac{13\sqrt{7}-34}{54} \right)^n = \frac{1}{\pi},$$

we have checked that if we write

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} \left[ (7\sqrt{7}-10) + (39\sqrt{7}-21)n \right] \left( \frac{13\sqrt{7}-34}{54} \right)^n = \alpha_p + \beta_p \sqrt{7},$$

then, for primes  $p > 7$ , we have the following supercongruences

$$\alpha_p \equiv -10 \left( \frac{-1}{p} \right) p \quad \beta_p \equiv 7 \left( \frac{-7}{p} \right) p \pmod{p^3}.$$

**Example 3.** Consider the Apéry sequence

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2, \quad n = 0, 1, 2, \dots$$

Sato's 2002 formula says that

$$\sum_{n=0}^{\infty} A_n \left[ (60\sqrt{15} - 134\sqrt{3}) + (72\sqrt{15} - 160\sqrt{3})n \right] \left( \frac{\sqrt{5}-1}{2} \right)^{12n} = \frac{1}{\pi}.$$

If we write

$$\sum_{n=0}^{p-1} A_n \left[ (60\sqrt{15} - 134\sqrt{3}) + (72\sqrt{15} - 160\sqrt{3})n \right] \left( \frac{\sqrt{5}-1}{2} \right)^{12n} = \alpha_p \sqrt{3} + \beta_p \sqrt{15},$$

then, for primes  $p > 5$ , we have the supercongruences

$$\alpha_p \equiv -134 \left( \frac{-3}{p} \right) p \quad \beta_p \equiv 60 \left( \frac{-15}{p} \right) p \pmod{p^3}.$$

**Example 4.** The Ramanujan-type series in [3, eq 6.1] is

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left[ (73 + 52\sqrt{2} - 42\sqrt{3} - 30\sqrt{6}) + (168 + 120\sqrt{2} - 96\sqrt{3} - 69\sqrt{6})n \right] \times \\ (-18872 - 13344\sqrt{2} + 10896\sqrt{3} + 7704\sqrt{6})^n = \frac{1}{\pi}.$$

It derives also from Ramanujan series [3, eq 6.4] by Zudilin's translation method. Writing

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left[ (73 + 52\sqrt{2} - 42\sqrt{3} - 30\sqrt{6}) + (168 + 120\sqrt{2} - 96\sqrt{3} - 69\sqrt{6})n \right] \times \\ (-18872 - 13344\sqrt{2} + 10896\sqrt{3} + 7704\sqrt{6})^n = \alpha_p + \beta_p \sqrt{2} + \gamma_p \sqrt{3} + \delta_p \sqrt{6},$$

we have, for primes  $p > 3$ , the supercongruences

$$\alpha_p \equiv 73 \left( \frac{-1}{p} \right) p \quad \beta_p \equiv 52 \left( \frac{-2}{p} \right) p \quad \gamma_p \equiv -42 \left( \frac{-3}{p} \right) p \quad \delta_p \equiv -30 \left( \frac{-6}{p} \right) p \pmod{p^3}.$$

**Example 5.** The "complex" Ramanujan series [7, eq. 45] is

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3} \left( \frac{7\sqrt{7} - 13\sqrt{-1}}{64} + \frac{15\sqrt{7} - 21\sqrt{-1}}{32} n \right) \left( \frac{47 + 45\sqrt{-7}}{128} \right)^n = \frac{1}{\pi}.$$

Writing

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^3}{(1)_n^3} \left[ (7\sqrt{7} - 13\sqrt{-1}) + (30\sqrt{7} - 42\sqrt{-1})n \right] \left( \frac{47 + 45\sqrt{-7}}{128} \right)^n = \alpha_p \sqrt{-1} + \beta_p \sqrt{7},$$

we have, for primes  $p > 7$ , the supercongruences

$$\alpha_p \equiv -13p \quad \beta_p \equiv 7 \left( \frac{-7}{p} \right) p \pmod{p^3}.$$

**Example 6.** For the Ramanujan-like series [8, eq. 10]

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{3})_n (\frac{2}{3})_n (\frac{1}{6})_n (\frac{5}{6})_n}{(1)_n^5} (-1)^n \left( \frac{3}{4} \right)^{6n} (45 + 549n + 1930n^2) = \frac{384}{\pi^2},$$

we have checked that if we write

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n (\frac{1}{3})_n (\frac{2}{3})_n (\frac{1}{6})_n (\frac{5}{6})_n}{(1)_n^5} (-1)^n \left( \frac{3}{4} \right)^{6n} (45 + 549n + 1930n^2) = \alpha_p,$$

then, for primes  $p > 3$ , we have the supercongruences

$$\alpha_p \equiv 45p^2 \pmod{p^5},$$

that is, they follow Zudilin's pattern [10].

**Example 7.** The only known (unproven) hypergeometric Ramanujan-like series for  $1/\pi^2$  with a non rational value of  $z$  is

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3 (\frac{1}{3})_n (\frac{2}{3})_n}{(1)_n^5} \left( \frac{15\sqrt{5} - 33}{2} \right)^{3n} \times \left[ (56 - 25\sqrt{5}) + (303 - 135\sqrt{5})n + (1220/3 - 180\sqrt{5})n^2 \right] = \frac{1}{\pi^2},$$

see [8, eq. 9]. Writing

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^3 (\frac{1}{3})_n (\frac{2}{3})_n}{(1)_n^5} \left( \frac{15\sqrt{5} - 33}{2} \right)^{3n} \times \left[ (56 - 25\sqrt{5}) + (303 - 135\sqrt{5})n + (1220/3 - 180\sqrt{5})n^2 \right] = \alpha_p + \beta_p \sqrt{5},$$

we have, for primes  $p > 5$ , the supercongruences

$$\alpha_p \equiv 56p^2 \quad \beta_p \equiv -25 \left( \frac{5}{p} \right) p^2 \pmod{p^5},$$

which generalizes Zudilin's pattern.

**More support.** To support even more our observations, we have considered many of the series, involving only simple square roots in [3], [4] and [5], and observed the expected mosaic supercongruences in all the cases.

#### 4. CONCLUDING REMARKS

An excellent survey on Ramanujan-type series is [2] and a beautiful survey on recent advances is in [10]. There are many examples of convergent Ramanujan-type and Ramanujan-Sato-type series in the literature, the most spectacular are of simple and very fast series. From the modular theory of Ramanujan-type series we know that there are functions  $z(q)$ ,  $b(q)$  and  $a(q)$ , with  $q = e^{i\pi\tau}$  and  $\text{Im}(\tau) > 0$ , which take algebraic values when  $\tau$  is a quadratic irrational. Obviously the series are faster as  $\text{Im}(\tau)$  is bigger. If  $\text{Im}(\tau)$  is small enough then the series is "divergent". An example of a "divergent" Ramanujan-type series is in [4, p. 371], which corresponds to  $\text{Im}(\tau) = \sqrt{253}/11$ . Convergent or divergent series lead to supercongruences following exactly the same patterns [7]. This fact is not very surprising if we observe that the sum from  $n = 0$  to  $p - 1$  do not discern the convergent or divergent origin.

Taking into account that the Jacobi symbols are the quadratic residues, perhaps it can give the clue to discover the pattern when the algebraic numbers involved are more complicated. We will continue investigating on this idea.

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E-mail address: [jguillera@gmail.com](mailto:jguillera@gmail.com)

AV. CESÁREO ALIERTA, 31 ESC. IZDA 4<sup>o</sup>–A, ZARAGOZA (SPAIN)